

A rapidly converging Ramanujan-type series for Catalan's constant

F. M. S. Lima

Institute of Physics, University of Brasília, P.O. Box 04455, 70919-970, Brasília-DF, Brazil

Abstract

In this note, by making use of a hypergeometric series identity derived by Guillera, I prove a Ramanujan-type series for the Catalan's constant. The convergence rate of this central binomial series representation surpasses those of all known similar series, including a classical formula by Ramanujan and a recent formula by Lupas. Interestingly, this suggests that an Apéry-like irrationality proof could be found for this constant.

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1. Introduction

Catalan's constant, so named in honor to Eugène C. Catalan (1814-1894), who first developed series and definite integrals representations for it, is a classical mathematical constant which may be defined as [5]

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941 \dots \quad (1)$$

This constant is a special value of some important functions such as the Dirichlet's beta function $\beta(s) := \sum_{n=0}^{\infty} (-1)^n / (2n+1)^s$, namely $\beta(2) = G$, and the Clausen's function $\text{Cl}_2(\theta) := \Im(\text{Li}_2(e^{i\theta}))$, namely¹

$$\text{Cl}_2\left(\frac{\pi}{2}\right) = G \quad (2)$$

and $\text{Cl}_2(3\pi/2) = -G$. For positive integer values of n , we can trace an analogy between $\beta(n)$ and $\zeta(n) := \sum_{k=1}^{\infty} 1/k^n$, the Riemann zeta function, since

Email address: `fabio@fis.unb.br` (F. M. S. Lima)

¹As usual, $\text{Li}_2(z)$ denotes the dilogarithm function, defined as $\sum_{n=1}^{\infty} z^n / n^2$ for real values of z , $z < 1$, and extended to \mathbb{C} , except for the cut $[1, \infty)$, by analytic continuation.

both $\zeta(2n)$ and $\beta(2n-1)$ are rational multiples of π^{2n} and π^{2n-1} , respectively, whereas finite closed-form expressions for both $\zeta(2n+1)$ and $\beta(2n)$ in terms of other basic constants are unknown [8]. However, the proof by Apéry (1978) that $\zeta(3)$ is irrational [2] has created an ‘asymmetry’ in that analogy because the irrationality of $\beta(2)$, though very suspected, remains unproven.²

From the point of view of numerical computation, Catalan himself (1865) computed G to 14 decimal places [4]. By making use of a technique from Kummer, Bresse (1867) computed it to 24 decimals, a result that was improved to 32 decimals by Glaisher (1913) [14]. With the advent of computers, G has been computed to a large number of digits. For instance, Yee and Chan (2009) computed it to 31 billion decimals [15]. Their computation employs two formulas, one of which is a central binomial formula due to Ramanujan (1915) [10]:³

$$G = \frac{\pi}{8} \ln(2 + \sqrt{3}) + \frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \binom{2n}{n}}. \quad (3)$$

On searching for similar rapidly converging series, Lupas (2000) has found the following alternating series [9]

$$G = -\frac{1}{64} \sum_{n=1}^{\infty} (-1)^n \frac{2^{8n} (40n^2 - 24n + 3)}{n^3 (2n-1) \binom{2n}{n} \binom{4n}{2n}^2}. \quad (4)$$

This series converges so fast that it has been implemented in *Mathematica*TM (version 6) for computing G .

On searching for new congruences modulo primes, Z.-W. Sun (2011) has pointed out that the following central binomial series should converge to G (see Conjecture A7 of Ref. [13]):

$$G \stackrel{?}{=} -\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{(3n-1) 8^n}{n^3 \binom{2n}{n}^3}. \quad (5)$$

He indeed comments that this formula could be derived from a hypergeometric identity proved by Guillera in a recent work [6], but a complete proof is not provided there in Ref. [13].

Here in this note, starting from a Guillera’s hypergeometric identity I prove a Ramanujan-type series representation for Catalan’s constant similar to that in Eq. (5) whose convergence rate surpasses that of all known central binomial series representations for this constant.

²Presently, the only known irrationality results for even beta values are the recent proofs by Rivoal and Zudilin (2003) that there exist infinitely many positive integers n for which $\beta(2n)$ is irrational, and that at least one of the seven numbers $\beta(2), \dots, \beta(14)$ is irrational [11].

³This can be proved from the fact that $G = -\int_0^{\pi/4} \ln(\tan \theta) d\theta = -\frac{3}{2} \int_0^{\pi/12} \ln(\tan \theta) d\theta$, as nicely described in Ref. [3].

2. A new Ramanujan-type series for G

Let us adopt the usual notation for the generalized hypergeometric series:

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}, \quad (6)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. Our main result makes use of the lemma below, which determines a special value for ${}_3F_2 \left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} ; z \right)$, a function that converges at $z = 1$ whenever $\Re\{(b_1 + b_2) - (a_1 + a_2 + a_3)\} > 0$ (see, e.g., Eq. (2.2.1) of Ref. [12]).

Lemma 1 (A special value).

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, 1, 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} ; 1 \right) = 2G.$$

PROOF. We start from a well-known integral representation for generalized hypergeometric functions (see, e.g., Eq. (1.2) of Ref [7]), namely

$${}_{p+1}F_p \left(\begin{matrix} \alpha, \alpha_1, \dots, \alpha_p \\ \gamma, \beta_1, \dots, \beta_{p-1} \end{matrix} ; t \right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 z^{\alpha-1} (1-z)^{\gamma-\alpha-1} {}_pF_{p-1} \left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_{p-1} \end{matrix} ; tz \right) dz, \quad (7)$$

valid whenever $\Re(\alpha) > 0$ and $\Re(\gamma - \alpha) > 0$. It then follows that

$$\begin{aligned} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, 1, 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} ; 1 \right) &= \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(1)} \int_0^1 z^{-\frac{1}{2}} (1-z)^0 {}_2F_1 \left(\begin{matrix} 1, 1 \\ \frac{3}{2} \end{matrix} ; 1z \right) dz \\ &= \frac{1}{2} \int_0^1 \frac{{}_2F_1 \left(\begin{matrix} 1, 1 \\ \frac{3}{2} \end{matrix} ; z \right)}{\sqrt{z}} dz \\ &= \int_0^1 {}_2F_1 \left(\begin{matrix} 1, 1 \\ \frac{3}{2} \end{matrix} ; x^2 \right) dx. \end{aligned} \quad (8)$$

Now, let us show that, for all $x \in (0, 1)$,

$${}_2F_1 \left(\begin{matrix} 1, 1 \\ \frac{3}{2} \end{matrix} ; x^2 \right) = \frac{\arcsin x}{x \sqrt{1-x^2}}. \quad (9)$$

It is well-known that

$${}_2F_1 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix} ; x^2 \right) = \frac{\arcsin x}{x} \quad (10)$$

for all non-null values of x for which the hypergeometric series at the left-hand side converges (see Eq. (1.5.10) of Ref. [12]). Two successive applications of the Euler transformation formula

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) = (1-z)^{-a} {}_2F_1 \left(\begin{matrix} a, c-b \\ c \end{matrix} ; \frac{z}{z-1} \right) \quad (11)$$

on Eq. (10) lead us to

$$\frac{\arcsin x}{x} = \frac{1}{\sqrt{1-x^2}} {}_2F_1\left(\frac{1}{2}, \frac{1}{\frac{3}{2}}; \frac{-x^2}{1-x^2}\right) \quad (12)$$

and, after some algebra,

$$\frac{\arcsin x}{x} = \frac{1-x^2}{\sqrt{1-x^2}} {}_2F_1\left(1, \frac{1}{\frac{3}{2}}; x^2\right). \quad (13)$$

This completes the proof of Eq. (9). From Eqs. (8) and (9), one has

$${}_3F_2\left(\frac{1}{2}, 1, \frac{1}{\frac{3}{2}}; 1\right) = \int_0^1 \frac{\arcsin x}{x \sqrt{1-x^2}} dx. \quad (14)$$

The trigonometric substitution $x = \sin \theta$ reduces this integral to

$${}_3F_2\left(\frac{1}{2}, 1, \frac{1}{\frac{3}{2}}; 1\right) = \int_0^{\pi/2} \frac{\theta}{\sin \theta} d\theta, \quad (15)$$

which can be solved in terms of the dilogarithm function $\text{Li}_2(z)$ as follows. First, note that

$$\int \frac{\theta}{\sin \theta} d\theta = \theta \ln \left[\frac{1 - \exp(i\theta)}{1 + \exp(i\theta)} \right] + i [\text{Li}_2(-e^{i\theta}) - \text{Li}_2(e^{i\theta})], \quad (16)$$

as can be easily checked by differentiating the right-hand side. Then

$$\begin{aligned} & \int_0^{\pi/2} \frac{\theta}{\sin \theta} d\theta \\ &= \frac{\pi}{2} \ln \left(\frac{1-i}{1+i} \right) + i [\text{Li}_2(-i) - \text{Li}_2(i)] - \left\{ \lim_{a \rightarrow 0^+} a \ln \left[\frac{1 - e^{(ia)}}{1 + e^{(ia)}} \right] + i [\text{Li}_2(-1) - \text{Li}_2(1)] \right\} \\ &= \frac{\pi}{2} \ln \left[\frac{(1-i)^2}{2} \right] + i \left[-2i \text{Cl}_2\left(\frac{\pi}{2}\right) \right] - \left\{ 0 + i \left(-\frac{\pi^2}{12} - \frac{\pi^2}{6} \right) \right\} \\ &= \frac{\pi}{2} \ln(-i) + 2 \text{Cl}_2\left(\frac{\pi}{2}\right) + i \frac{\pi^2}{4} = \frac{\pi}{2} \left(\ln 1 - i \frac{\pi}{2} \right) + 2G + i \frac{\pi^2}{4} \\ &= 2G, \end{aligned} \quad (17)$$

where the special value of the Clausen function in Eq. (2) and the principal value of the logarithm function, with $\text{Arg}(z) \in (-\pi, \pi]$, were taken into account.⁴ The substitution of this result in Eq. (15) completes the proof.

□

⁴On Entry 9 of Adamchik's webpage [1], where several representations for G are proved computationally with *Mathematica*TM, one finds the integral formula $\frac{1}{2} \int_0^{\pi/2} \theta / \sin \theta d\theta = G$. Our Eqs. (16) and (17) can then be viewed as a formal proof of this formula.

We are now in a position to prove a rapidly converging central binomial formula for the Catalan's constant, which is our main result.

Theorem 1 (Rapidly converging central binomial series for G).

$$G = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(3n+2) 8^n}{(2n+1)^3 \binom{2n}{n}^3}.$$

PROOF. Let

$$f(x) := \sum_{n=0}^{\infty} (-1)^n \frac{(x + \frac{1}{2})_n^3}{8^n (x+1)_n^3} [6(x+n)+1] \quad (18)$$

be a function of a real variable x in the open interval $(0, 1)$. Then

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \sum_{n=0}^{\infty} (-1)^n \frac{(1)_n^3 (6n+4)}{8^n \left(\frac{3}{2}\right)_n^3} = \sum_{n=0}^{\infty} (-1)^n \frac{n!^3 (6n+4)}{8^n (2n+1)^3 \frac{(2n)!^3}{n!^3 4^{3n}}} \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \frac{n!^6 (3n+2) 64^n}{8^n (2n+1)^3 (2n)!^3} \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \frac{(3n+2) 8^n}{(2n+1)^3 \binom{2n}{n}^3}. \end{aligned} \quad (19)$$

On the other hand, from Guillera's third identity in Ref. [6], we know that

$$f(x) = 4x \sum_{n=0}^{\infty} \frac{(x/2 + \frac{1}{4})_n (x/2 + \frac{3}{4})_n}{(x+1)_n^2} \quad (20)$$

for all values of x for which this series converges. Therefore

$$f(x) = 4x {}_3F_2\left(\frac{2x+1}{4}, \frac{2x+3}{4}, 1; x+1, x+1; 1\right), \quad (21)$$

which implies that

$$f\left(\frac{1}{2}\right) = 2 {}_3F_2\left(\frac{1}{2}, 1, 1; \frac{3}{2}, \frac{3}{2}; 1\right). \quad (22)$$

On substituting the result obtained in Lemma 1 for ${}_3F_2\left(\frac{1}{2}, 1, 1; \frac{3}{2}, \frac{3}{2}; 1\right)$, we find $f(\frac{1}{2}) = 4G$. From Eq. (19), one then has

$$2 \sum_{n=0}^{\infty} (-1)^n \frac{(3n+2) 8^n}{(2n+1)^3 \binom{2n}{n}^3} = 4G, \quad (23)$$

which completes the proof. \square

The convergence rate of the just proved central binomial series representation for G is to be compared to that of other known similar series, including those in Eqs. (3) and (4). This is done in details in the next section.

3. Convergence rates

Let us now check the convergence rate of each Ramanujan-type series for G mentioned in this work. By applying the Stirling's improved formula, namely

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi \left(n + \frac{1}{6}\right)}, \quad (24)$$

which implies that

$$\binom{2n}{n} \sim \frac{2^{2n} \sqrt{2n + \frac{1}{6}}}{\sqrt{2\pi} \left(n + \frac{1}{6}\right)}, \quad (25)$$

we shall develop an order estimate of the n -th term for each series.

We begin with Ramanujan's series, given in Eq. (3). Its n -th term is

$$\begin{aligned} \frac{1}{(2n+1)^2 \binom{2n}{n}} &\sim \frac{\sqrt{2\pi \left(n + \frac{1}{6}\right)}}{2^{2n} \sqrt{2n + \frac{1}{6}} (2n+1)^2} \\ &= \sqrt{\frac{\pi}{3}} \frac{6n+1}{2^{2n} \sqrt{12n+1} (2n+1)^2} \sim \frac{6n+1}{2^{2n} \sqrt{12n+1} (2n+1)^2} \\ &\sim \frac{3}{2^{2n} (2n+1) \sqrt{12n+1}}. \end{aligned} \quad (26)$$

The more complex central binomial series by Lupas, see Eq. (4), has an n -th term whose absolute value can be estimated as follows:

$$\begin{aligned} \frac{2^{8n} |40n^2 - 24n + 3|}{n^3 (2n-1) \binom{2n}{n} \binom{4n}{2n}^2} &\sim \frac{2^{8n} |40n^2 - 24n + 3|}{n^3 (2n-1) \frac{2^{2n} \sqrt{2n + \frac{1}{6}}}{\sqrt{2\pi} \left(n + \frac{1}{6}\right)} 2^{8n} \frac{3}{\pi} \frac{24n+1}{(12n+1)^2}} \\ &\sim \frac{|40n^2 - 24n + 3| (12n+1)^2 \sqrt{2\pi} \left(n + \frac{1}{6}\right)}{n^3 (2n-1) 2^{2n} \sqrt{2n + \frac{1}{6}} (24n+1)}. \end{aligned} \quad (27)$$

For sufficiently large values of n , this simplifies to

$$\begin{aligned} \frac{(40n-24) (12n+1)^2 \sqrt{2\pi} \left(n + \frac{1}{6}\right)}{2^{2n} n^2 (2n-1) \sqrt{2n + \frac{1}{6}} (24n+1)} &\sim \frac{(40n-24) (12n+1)^2 (6n+1)}{2^{2n} n^2 (2n-1) \sqrt{12n+1} (24n+1)} \\ &= \frac{4 (5n-3) (12n+1)^2 (6n+1)}{2^{2n} n^2 (2n-1) \sqrt{12n+1} (12n + \frac{1}{2})} \sim \frac{4 (5n-3) (12n+1) (6n+1) \frac{5}{2}}{2^{2n} n^2 (5n - \frac{5}{2}) \sqrt{12n+1}} \\ &\sim \frac{10 (12n+1) (6n+1)}{2^{2n} n^2 \sqrt{12n+1}} = \frac{10 \sqrt{12n+1} (6n+1)}{2^{2n} n^2} \\ &= \frac{5 \sqrt{12n+1} (6n+1)}{2^{2n-1} n^2}. \end{aligned} \quad (28)$$

This convergence rate is clearly slower than that of Ramanujan's series, not to say the number of basic arithmetic operations needed to compute the n -th term, which is considerably larger in the Lupaş' series.⁵

The convergence rate of the series we proved in our Theorem 1 can be estimated as follows. The n -th term of our series is

$$\begin{aligned} \frac{(3n+2) 2^{3n}}{(2n+1)^3 \binom{2n}{n}^3} &\sim \frac{(3n+2) 2^{3n}}{(2n+1)^3 \frac{2^{6n+3}}{(12n+2)^{\frac{3}{2}}}} \\ &= \frac{3(n+\frac{2}{3}) 2^{3n} (12n+2)^{\frac{3}{2}}}{8(n+\frac{1}{2})^3 2^{6n+3}}. \end{aligned} \quad (29)$$

For sufficiently large values of n , this can be approximated by

$$\begin{aligned} \frac{3}{8} \frac{(12n+2)^{\frac{3}{2}}}{(n+\frac{1}{2})^2 2^{3n+3}} &= 3\sqrt{2} \frac{(6n+1)^{\frac{3}{2}}}{(2n+1)^2 2^{3n+3}} \\ &\sim 9\sqrt{2} \frac{\sqrt{6n+1}}{(2n+1) 2^{3n+3}} = 27\sqrt{2} \frac{\sqrt{6n+1}}{(\sqrt{6n+3})^2 2^{3n+3}} \\ &\sim 27\sqrt{2} \frac{1}{\sqrt{6n+3} 2^{3n+3}} \sim \frac{5/\sqrt{3}}{2^{3n} \sqrt{2n+1}}. \end{aligned} \quad (30)$$

The factor 2^{3n} makes our series converges faster than both the Ramanujan and Lupaş' series.

In Table 1, below, we compare the error committed in approximating G by the partial sum of the first N terms corresponding to each central binomial series mentioned in this work. It is clear from this table that our series is the one that yields the smallest absolute error. The only competitive series is that conjectured by Sun, see our Eq. (5), but a direct comparison of its n -th term with that of the series in our Theorem 1 shows that it converges slower. Therefore, even an eventual proof of Sun's conjecture will not furnish a central binomial series faster than the one I am presenting here in this work.

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⁵Despite these disadvantages, Lupaş' series has been implemented in *Mathematica*TM (version 6) for computing G [14].

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Tables

N	Lupas	Ramanujan	Sun	Theorem 1
5	$+2.9 \times 10^{-4}$	-3.0×10^{-6}	$+1.1 \times 10^{-5}$	-1.3×10^{-6}
10	-2.0×10^{-7}	-1.3×10^{-9}	-2.6×10^{-10}	$+3.1 \times 10^{-11}$
50	-7.7×10^{-32}	-1.1×10^{-34}	-9.1×10^{-47}	$+1.1 \times 10^{-47}$
100	-4.3×10^{-62}	-3.3×10^{-65}	-4.5×10^{-92}	$+5.6 \times 10^{-93}$
500	-2.9×10^{-303}	-4.6×10^{-307}	-1.2×10^{-453}	$+1.5 \times 10^{-454}$
1000	-1.9×10^{-604}	-1.5×10^{-608}	-2.4×10^{-905}	$+2.9 \times 10^{-906}$

Table 1: Deviations from G of the partial sums obtained by adding the first N terms of each central binomial series mentioned in the text.